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COMMENT

On the universality class of Ising models with mixed two- and three-spin interactions

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Abstract. It is shown that the Hamiltonian limit of a spin-$\frac{1}{2}$ Ising model on a D-dimensional lattice with three-spin interactions in one direction and two-spin interactions in the other directions is equivalent to the same limit of a nearest-neighbour four-state spin model.

Recently, models with many-body interactions have attracted a growing interest. The critical properties of these models generally depend on the range of the interaction and the order of the transition turns from second order to first order by increasing the range of the interaction (Turban 1982, Penson *et al* 1982, Maritan *et al* 1984, Iglói *et al* 1986, 1987, Alcaraz 1986). There are a few cases known when the universality class of a multispin coupling model and that of a nearest-neighbour coupling model are the same. Several models belong to the $D=2$ four-state Potts universality class. A well known example is the Baxter-Wu model (Baxter and Wu 1973). A further example with an Ising model with mixed two-spin and three-spin interactions on a special two-dimensional lattice was constructed by Horiguchi and Gonçalves (1985). A similar model on a square lattice introduced by Debierre and Turban (1983) also belongs to the four-state Potts universality class. This was shown recently by Blöte *et al* (1986) using mapping between the two models.

The later model has been generalised very recently by Alcaraz and Barber (1986) by introducing sublattices and defining different couplings. The model according to their numerical results shows very similar critical behaviour to the Ashkin-Teller model. Our aim in this comment is to show that the critical properties of the two models are exactly the same. To justify this statement we construct a mapping between the two models following the method of Blöte *et al* (1986).

To show the equivalence of the two systems a somewhat more general Ising model is treated on a D-dimensional lattice than that introduced by Alcaraz and Barber (1986). This lattice consists of $(D-1)$-dimensional planes being perpendicular to the t direction. Furthermore, the lattice is divided into three sublattices, containing the spins of the $(3i+1)$th, $(3i+2)$th and $(3i+3)$th planes ($i=0,1,\ldots$), respectively. Within one plane there are two-spin interactions, while in the t direction three-spin interactions exist. The Hamiltonian of the system is

$$-\beta H=\sum_{j,i}(H^{p}_{3i,j}+H^{t}_{3i,j}) \tag{1}$$

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where

$$H^{p}_{i,j}=\tfrac{1}{2}\sum_{j'=1}^{d}(K^{1}_{2}\sigma(i+1,j)\sigma(i+1,j')+K^{2}_{2}\sigma(i+2,j)\sigma(i+2,j')$$
$$+K^{3}_{2}\sigma(i+3,j)\sigma(i+3,j')) \qquad (2)$$

with $\sigma=\pm1$ and (i,j) denotes the position of a spin in the ith plane, while its nearest neighbours in this plane are characterised by the coordinates (i,j'), $(j'=1,2,\ldots,d)$. The interaction in the t direction is

$$H^{t}_{i,j}=K^{1}_{3}(\sigma(i,j)\sigma(i+1,j)\sigma(i+2,j)-1)+K^{2}_{3}(\sigma(i+1,j)\sigma(i+2,j)\sigma(i+3,j)-1)$$
$$+K^{3}_{3}(\sigma(i+2,j)\sigma(i+3,j)\sigma(i+4,j)-1). \qquad (3)$$

Now let us build up the transfer matrix (T) of the model in the t direction. Since in this direction there are three-spin interactions the transfer matrix connects the states of two planes with the next two. T itself has a complicated structure, but as was observed in a similar problem by Blöte *et al* (1986) the $T^{3/2}$ matrix has a simple form in the very anisotropic limit. This time-continuum limit (Kogut 1979) can be reached when the different couplings satisfy the following relations:

$$\begin{aligned}&\exp(-2K^{1}_{3})=\tau h_1 \qquad \exp(-2K^{2}_{3})=\tau h_2 \qquad \exp(-2K^{3}_{3})=\tau h_3\\ &K^{1}_{2}=\tau\lambda_1 \qquad K^{2}_{2}=\tau\lambda_2 \qquad K^{3}_{3}=\tau\lambda_3\end{aligned} \qquad (4)$$

where τ is an infinitesimal number and $T^{3/2}$ is

$$T^{3/2}=1-\tau H+\mathrm{O}(\tau^2). \qquad (5)$$

In the following we construct the Hamilton operator H. The $T^{3/2}$ matrix connects the states of three planes (called vectors in the following) with the next three ones, and only those matrix elements are of $\mathrm{O}(\tau)$ when the two vectors differ no more than in the states of one triplet of spins. Thus in $\mathrm{O}(\tau)$ the contributions are only from the diagonal and from the one-spin-flip terms. However, due to the presence of three-spin interactions some of these terms are also of $\mathrm{O}(\tau^2)$. Now let us first determine the $\mathrm{O}(\tau)$ matrix elements of the three-spin interaction terms of the Hamiltonian (equation (3)). These are presented in table 1.

As one can see from this table, only one-half of the diagonal terms and one-quarter of the one-spin-flip processes are of $\mathrm{O}(\tau)$. Furthermore, the different processes may be uniquely characterised by the states of the last two spins in each triplet, since the first spins are always determined by the condition that the corresponding matrix element is at least of $\mathrm{O}(\tau)$. Now let us identify the configurations of the last two spins: $++$, $+-$, $--$ and $-+$ with the states $|1\rangle$, $|2\rangle$, $|3\rangle$ and $|4\rangle$, respectively, of a four-state spin variable denoted by S. With this variable the matrix elements of the two-spin interaction

Table 1. Matrix elements of the $T^{3/2}$ matrix associated with the three-spin interactions. First column: order O(1); second and third columns: order $\mathrm{O}(\tau)$.

$+++\mid+++$	$+++\mid++-$	$-+-\mid-++$
$-+-\mid-+-$	$+++\mid+--$	$+--\mid+++$
$+--\mid+--$	$+++\mid--+$	$--+\mid+++$
$--+\mid--+$	$-+-\mid+--$	$+--\mid-+-$
	$-+-\mid--+$	$--+\mid-+-$
	$+--\mid+-+$	$--+\mid---$

terms of the Hamiltonian (equation (2)) can be obtained in a simple form in $O(\tau)$ as well. (This is not true when more spins are coupled in the t direction.) Collecting all terms in order τ the Hamilton operator H in equation (5) has the following form:

$$H=\tfrac{1}{2}\sum_{j,j'} E(S_j, S_{j'})-\sum_{j} M_j. \tag{6}$$

Here j and j' are nearest-neighbour spins in the $(D-1)$-dimensional plane. The diagonal term of this operator is the sum of the interaction energies between neighbouring spins:

$$\begin{aligned}
&E(1,1)=E(2,2)=E(3,3)=E(4,4)=-\lambda_1-\lambda_2-\lambda_3\\
&E(1,2)=E(2,1)=E(3,4)=E(4,3)=\lambda_1+\lambda_2-\lambda_3\\
&E(1,3)=E(3,1)=E(2,4)=E(4,2)=\lambda_1-\lambda_2+\lambda_3\\
&E(1,4)=E(4,1)=E(2,3)=E(3,2)=-\lambda_1+\lambda_2+\lambda_3
\end{aligned} \tag{7}$$

while the spin-flip operator is given as

$$M=\begin{pmatrix} 0 & h_1 & h_2 & h_3\\ h_3 & 0 & h_1 & h_2\\ h_2 & h_1 & 0 & h_3\\ h_1 & h_2 & h_3 & 0 \end{pmatrix}. \tag{8}$$

This Hamilton operator describes an equivalent $(D-1)$-dimensional quantum system to the original model.

It is easy to show, however, that this quantum system has another equivalent D-dimensional statistical mechanical system. Let us consider a four-state spin model defined on the same D-dimensional lattice as the original one. The interaction energy between nearest-neighbour spins has the same form as given in equation (7), but the coupling constants are K_δ^1/β, K_δ^2/β, K_δ^3/β instead of λ_1, λ_2 and λ_3, respectively, and $\delta=t$ in the t direction and $\delta=x$ in all the other directions. Taking the time-continuum limit in the following way:

$$\begin{aligned}
&\exp(-2K_t^1-2K_t^2)=\tau h_1 \qquad &&\exp(-2K_t^1-2K_t^3)=\tau h_2\\
&\exp(-2K_t^2-2K_t^3)=\tau h_3 \qquad &&K_x^1=\tau\lambda_1,\ K_x^2=\tau\lambda_2,\ K_x^3=\tau\lambda_3
\end{aligned} \tag{9}$$

one gets back the Hamiltonian in equation (6). Now supposing that anisotropy does not affect the critical properties of systems then the critical behaviour of the two models are the same.

In the following let us briefly discuss some special cases of this mapping. When in the Hamiltonian system $h_1=h_2=h_3$ and $\lambda_1=\lambda_2=\lambda_3$, then the two equivalent systems are the D-dimensional four-state Potts model and the D-dimensional Ising model given in equation (1), when the three sublattices are equivalent. It has already been shown by Blöte *et al* (1986) that the two models for $D=2$ are in the same universality class. Recent numerical investigations support the validity of this mapping (Alcaraz and Barber 1987, Vanderzande and Iglói 1987). A further result is that, according to the above mapping, the transition in this Ising model for $D\geqslant 3$ is of first order.

Another special case may be obtained when $h_1=h_2$ and $\lambda_1=\lambda_2$, which describes an Ising system equivalent to the Ashkin–Teller model. By choosing the subspace proposed by Kohmoto *et al* (1981), $h_1=h_3=1$, $h_2=\lambda$, $\lambda_1=\lambda_3=\beta$, $\lambda_2=\lambda\beta$, the equivalent $D=2$ Ising model is the same as that investigated recently by Alcaraz and Barber (1986). Thus the mapping presented in this comment explains the numerical findings that the two models show very similar critical behaviour.

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